AN APPLICATION OF FINITE DIFFERENCE METHOD TO SOLVE ELLIPTIC PDE'S WITH USE OF GS AND SOR ITERATIVE METHODS

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ABSTRACT.

1.1.1 In this paper finding an approximation solution of a linear system $Au = B$. By the Gauss Seidel (GS) and the Successive Over-Relaxation (SOR) iterative methods, this system was derived from an elliptic partial differential equation by adopting the finite difference method. These methods are applied to two different examples. The results showed that the convergent to the approximate solution of them with special choice of the parameter ω for the SOR iterative method and compared the results.

Keywords.

Poisson's Equation, Laplace's Equation, Approximation solution, finite difference method (FDM), GS and SOR methods.

1.1.2 I. Introduction

Solving Partial Differential Equations (PDEs) by analytically methods is not always available, numerically methods will be used, the FDM is one of the simple and powerful techniques to solve initial and boundary value problems for linear and nonlinear PDEs. In this paper, applied the Finite Difference Method (FDM) to solve the Elliptic PDEs, (Poisson and Laplace equations). This paper is

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organized as follows: in section one, the description of the Laplace and Poisson equations; in section two, the concept of using the technique of FDM; in section three, the derivation of all kinds of approximate difference formulas; and in section four, implementations for all kinds of equations are discussed.

II. The Elliptic Partial Differential Equations.

The Poisson equation is an elliptic partial differential equation and presented as; [7]

$$\nabla^2 u \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y) \qquad (1)$$

$for\ (x,y) \in \Omega$, and

$$u(x,y) = g(x,y) \quad for\ (x,y) \in \partial\Omega$$

where,

$$\Omega = \{(x,y): a < x < b\,, c < y < d\}.$$

And, $\partial\Omega$ denotes the boundary of Ω, the boundary values for the functions f and g are known at all points on the sides of rectangular Ω.It is used by French mathematician Simeon Poisson (1781-1840), assume that both functions f and g are continuous on their domain to ensure a unique solution is exist.

The Laplace equation is a special case of Poisson's equation present as follow.

$$\nabla^2 u \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \quad for\ (x,y) \in \Omega \qquad (2)$$

Both kinds of elliptic equation presented above can be solved by using numerical technique called the finite difference method.

III. The Finite Difference Method

Firstly partitioned the intervals [a,b] and [c,d] into n and m equal parts respectively by choosing integers n and m and define step size $h = \frac{b-a}{n}$ $and\ k = \frac{d-c}{m}$. The idea of finite difference is to start at the Taylor expression of the solution function $u(x_k)$ as:[3]

$$u(x_k + h) = u(x_k) + \frac{h}{1!}u'(x_k) + \frac{h^2}{2!}u''(x_k) + \frac{h^3}{3!}u'''(x_k) + o(h^4) \qquad (3)$$

$$u(x_k - h) = u(x_k) - \frac{h}{1!}u'(x_k) + \frac{h^2}{2!}u''(x_k) - \frac{h^3}{3!}u'''(x_k) + o(h^4) \quad (4)$$

Eqns. (3 and 4) forward and backward difference formula So the forward and backward differences formulas for the finite derivative are;

$$u'(x_k) = \frac{u(x_k + h) - u(x_k)}{h} + o(h) \quad (5)$$

$$u'(x_k) = \frac{u(x_k) - u(x_k - h)}{h} + o(h) \quad (6)$$

By subtracting Eqns. (4 and 3) gets on the central difference formula for the first derivative as;

$$u''(x_k) = \frac{u(x_k + h) - u(x_k - h)}{2h} + o(h^2) \quad (7)$$

By the sum of Eqns. (3 & 4) gets on the central difference formula for the second derivative as:

$$u''(x_k) = \frac{u(x_k + h) - 2u(x_k) + u(x_k - h)}{h^2} + o(h^2) \quad (8)$$

where $o(h^2) \to 0 \quad as \quad h \to 0.$

IV. Deriving an approximate difference formulas

Finite Difference formulation for Laplacian Differential Equation

May be rewriting in the following formula

$$\nabla^2 u \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \quad for\ (x,y) \in \Omega$$

And

$$u(x,y) = g(x,y) \quad forall\ (x,y) \in \partial\Omega$$

where

$$\Omega = \{(x,y): a < x < b, c < y < d\}$$

By choosing the integers n and m, and define step size $h = \frac{b-a}{n}$, $k = \frac{d-c}{m}$. Placing a grid on rectangle Ω by drawing vertical and horizontal lines through the points

$$x_i = a + ih, \qquad i = 0,1,\dots,n$$

$y_j = c + jk\,, \qquad j = 0,1, \dots, m$

x_i and y_i are called grid lines, and their intersections are called the mesh points of the grid

The formula for approximating $u''(x)$ is obtained from

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + o(h^2)$$

Applying Taylor series to generate the central difference formula

$$\frac{\partial^2 u(x_i, y_j)}{\partial x^2} = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + o(h^2) \qquad (9)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial y^2} = \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} + o(k^2) \qquad (10)$$

By substitution Eqns. (9 & 10) into The Laplacian Equation gets on;

$$\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{k^2} + o(h^2 + k^2) = 0$$

For each $i = 1, \dots, n-1, \quad j = 1, \dots, m-1$

By using the approximate $w_{i,j}$ for u(x_i, y_j) , so the previous equation rewritten in the form

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right] u_{i,j} - u_{i+1,j} - u_{i-1,j} - \left(\frac{h}{k}\right)^2 [u_{i,j+1} + u_{i,j-1}] = 0 \qquad (11)$$

For each $i = 1, \dots, n-1, \quad j = 1, \dots, m-1$

The boundary conditions are

$u(0, y_j) = g(x_0, y_j), \quad j = 0,1, \dots, m$

$u(x_n, y_j) = g(x_n, y_j), \quad j = 0,1, \dots, m$

$u(x_i, 0) = g(x_i, 0), \quad i = 1, \dots, n-1$

$u(x_i, y_m) = g(x_i, y_m), \quad i = 1, \dots, n-1$

by rewritten in the following forms

$u_{0,j} = g(x_0, y_j), \quad j = 0,1, \dots, m$

$u_{n,j} = g(x_n, y_j), \quad j = 0,1, \dots, m$

$u_{i,0} = g(x_i, 0), \quad i = 1, \dots, n-1$

$u_{i,m} = g(x_i, y_m), \quad i = 1, \dots, n-1$

By appling the Laplace computational formula (11) at each of the interior points of Ω will create a linear system of order $(n-1)(m-1)$

Finite Difference formulation for Poisson Differential Equation

consider the Poisson equation:

$$\nabla^2 u \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y) \quad for\ (x,y) \in \Omega$$

And $u(x,y) = f(x,y) \quad forall\ (x,y) \in \partial\Omega$

where $\Omega = \{(x,y): a < x < b, c < y < d\}$

By using the same procedure mentioned in the finite difference formulation for Laplacian equation with notation $f_{i,j} = f(x_i, y_j)$, and using the approximate $w_{i,j}$ for u(x_i, y_j) The equation can be written

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right] u_{i,j} - u_{i+1,j} - u_{i-1,j} - \left(\frac{h}{k}\right)^2 [u_{i,j+1} + u_{i,j-1}] = -h^2 f_{i,j}$$

For each $i = 1, \dots, n-1, \quad j = 1, \dots, m-1.$

And,

$u_{0,j} = g(x_0, y_j), \quad j = 0,1, \dots, m$

$u_{n,j} = g(x_n, y_j), \quad j = 0,1, \dots, m$

$u_{i,0} = g(x_i, 0), \quad i = 1, \dots, n-1$

$u_{i,m} = g(x_i, y_m), \quad i = 1, \dots, n-1$

V. The Gauss– Seidel (GS) and Successive Over–Relaxation (SOR) Methods

The form of Gauss-Seidel iteration is:[1, 2, 9]

$$X_i^{k+1} = \frac{b_i -, \sum_{j=1}^{i-1} a_{ij} X_j^{k+1} -, \sum_{j=i+1}^{n} a_{ij} X_j^{k}}{a_{ij}},$$

for each $i = 1,2, \dots, n \quad, k = 0,1,2, \dots, n$

The form of Successive Over-Relaxation SOR iteration is:[7]

$$X_i^{k+1} = (1-\omega)X_i^k + \omega\left[\frac{b_i -, \sum_{j=1}^{i-1} a_{ij} X_j^{k+1} -, \sum_{j=i+1}^{n} a_{ij} X_j^k}{a_{ij}}\right]$$

for each $i = 1,2,\dots,n \ ,k = 0,1,2,\dots,n$

Theorem

If A is strictly diagonal dominant, then for any choice of x_0 both Jacobi (J) and Gauss – Seidel (GS) methods give sequences $\{x_k\}_{k=0}^{\infty}$ that convergence to the solution of $Ax = B$. [4,7]

Theorem (Ostrowski-Reich)

If A is a positive definite matrix and $0 < \omega < 2$ then the successive over-relaxation (SOR) method converges for any choice of initial approximation solution vector x_0.

The SOR method used the iteration formula:

$$\omega = \frac{4}{2 + \sqrt{4 - \left[\cos\left(\frac{\pi}{n}\right) + \cos\left(\frac{\pi}{m}\right)\right]^2}}$$

where the parameter ω lies in the range $1 \le \omega < 2$. [4,7]

VI. Numerical Results

The results of numerical tests of the Gauss Seidel method (GS) and the successive over-relaxation (SOR) methods Are presented in the following examples

Example 1

Use GS and SOR iterative methods to compute the approximate solution for Laplace's equation presented in the equation

$$\nabla^2 u \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \quad for\ all\ (x,y) \in \Omega$$

$$\Omega = \{(x,y): 0 < x < 2, 0 < y < 2\}$$

at the boundary values:

$$u(x,0) = x^4, \quad u(x,2) = x^4 - 13.5x^2 + 5.0625 \quad for\ all \quad 0 \le x \le 2.$$

$$u(0,y) = y^4, \quad u(2,y) = y^4 - 13.5y^2 + 5.0625 \quad for\ all \quad 0 \le y \le 2.$$

Where $h = k = 0.5$

Solution

The finite difference equation is:

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j-1} - u_{i,j+1} = 0$$

For each $i = 1,2,3 \quad j = 1,2,3$

$p_1 \quad : \quad 4u_1 - u_2 - u_4 = u_{0,3} + u_{1,4}$

$p_2 \quad : \quad 4u_2 - u_1 - u_3 - u_5 = u_{2,4}$

$p_3 \quad : \quad 4u_3 - u_2 - u_6 = u_{3,4} + u_{4,3}$

$p_4 \quad : \quad 4u_4 - u_1 - u_5 - u_7 = u_{0,1}$

$p_5 \quad : \quad 4u_5 - u_2 - u_4 - u_6 - u_8 = 0$

$p_6 \quad : \quad 4u_6 - u_3 - u_5 - u_9 = u_{4,2}$

$p_7 \quad : \quad 4u_7 - u_4 - u_8 = u_{0,1} + u_{1,0}$

$p_8 \quad : \quad 4u_8 - u_5 - u_7 - u_9 = u_{2,0}$

$p_9 \quad : \quad 4u_9 - u_6 - u_8 = u_{3,0} + u_{4,1}$

The boundary conditions are:

$u_{0,j} = g(x_0, y_j) = y_j^4 \quad j = 0,1,\dots,4$

$u_{i,0} = g(x_i, 0) = x_i^4 \quad i = 1,\dots,3$

$u_{4,j} = g(x_4, y_j) = y_j^4 - 13.5y_j^2 + 5.0625 \quad j = 0,1,\dots,4$

$u_{i,4} = g(x_i, y_4) = x_i^4 - 13.5x_i^2 + 5.0625 \quad i = 1,\dots,3$

So, the linear system to this example has the form:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 6.8125 \\ -7.4375 \\ -40.5 \\ 0.0625 \\ 0 \\ -7.4375 \\ 0.125 \\ 1 \\ 6.8125 \end{bmatrix}$$

The result for GS iterative method is in the following tables:

Table 1 The approximate solution of Laplace's Equation using GS method

t	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
0	0	0	0	0	0	0	0	0	0
1	27.2500	-22.9375	-167.7344	7.0625	6.8125	-69.9805	2.2656	6.2695	11.3223
2	23.2813	-64.1602	-195.5352	8.3398	-10.1074	-78.3301	4.1523	5.3418	9.0029
3	13.2949	-77.8369	-201.0417	2.0850	-14.9233	-81.4905	2.3567	3.1091	7.6546
4	8.3120	-81.6633	-202.7885	-0.8137	-17.5174	-82.9128	1.0739	1.8028	6.9725
5	6.6308	-83.1688	-203.5204	-2.2032	-18.6198	-83.5419	0.3999	1.1881	6.6616
6	5.9070	-83.8083	-203.8376	-2.8282	-19.1117	-83.8219	0.0900	0.9100	6.5220
7	5.5909	-84.0896	-203.9779	-3.1077	-19.3303	-83.9465	-0.0494	0.7856	6.4598
8	5.4507	-84.2144	-204.0402	-3.2323	-19.4276	-84.0020	-0.1117	0.7301	6.4320
9	5.3883	-84.2699	-204.0680	-3.2877	-19.4709	-84.0267	-0.1394	0.7054	6.4197
10	5.3606	-84.2946	-204.0803	-3.3124	-19.4902	-84.0377	-0.1517	0.6944	6.4142
11	5.3483	-84.3056	-204.0858	-3.3234	-19.4988	-84.0426	-0.1572	0.6895	6.4117
12	5.3428	-84.3105	-204.0883	-3.3283	-19.5026	-84.0448	-0.1597	0.6874	6.4106
13	5.3403	-84.3126	-204.0894	-3.3305	-19.5043	-84.0457	-0.1608	0.6864	6.4102
14	5.3392	-84.3136	-204.0898	-3.3315	-19.5050	-84.0462	-0.1613	0.6860	6.4099
15	5.3387	-84.3140	-204.0901	-3.3319	-19.5054	-84.0464	-0.1615	0.6858	6.4099
16	5.3385	-84.3142	-204.0901	-3.3321	-19.5055	-84.0465	-0.1616	0.6857	6.4098
17	5.3384	-84.3143	-204.0902	-3.3322	-19.5056	-84.0465	-0.1616	0.6857	6.4098
18	5.3384	-84.3143	-204.0902	-3.3322	-19.5056	-84.0465	-0.1616	0.6856	6.4098
19	5.3384	-84.3144	-204.0902	-3.3322	-19.5056	-84.0465	-0.1616	0.6856	6.4098
20	5.3384	-84.3144	-204.0902	-3.3322	-19.5056	-84.0465	-0.1617	0.6856	6.4098

The result for SOR iterative method is in the following tables:

Table 2 The approximate solution of Laplace's Equation using SOR method

t	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
0	0	0	0	0	0	0	0	0	0
1	34.3245	-26.6646	-212.4546	11.1238	10.8089	-100.9726	4.1328	9.7437	5.5962
2	20.5195	-87.5884	-208.2798	8.5938	-25.0729	-82.9809	5.3314	-1.9456	6.1280
3	4.1217	-86.9200	-203.4875	-6.8349	-18.9364	-84.0429	-3.5193	0.4019	6.3947
4	3.7307	-83.7751	-204.0757	-3.8072	-20.2479	-84.2814	0.4711	0.7201	6.3506
5	5.7760	-84.5457	-204.2408	-3.1056	-19.2382	-83.9674	-0.2437	0.7164	6.4597
6	5.2233	-84.2538	-204.0071	-3.3690	-19.5767	-84.0475	-0.1422	0.6771	6.3938
7	5.3757	-84.3145	-204.1122	-3.3272	-19.4784	-84.0496	-0.1678	0.6894	6.4141
8	5.3302	-84.3152	-204.0858	-3.3295	-19.5151	-84.0459	-0.1580	0.6842	6.4084
9	5.3411	-84.3149	-204.0913	-3.3339	-19.5026	-84.0465	-0.1636	0.6859	6.4102
10	5.3370	-84.3141	-204.0898	-3.3319	-19.5068	-84.0466	-0.1610	0.6856	6.4096
11	5.3389	-84.3145	-204.0904	-3.3323	-19.5052	-84.0465	-0.1619	0.6856	6.4098
12	5.3381	-84.3143	-204.0901	-3.3322	-19.5058	-84.0465	-0.1616	0.6856	6.4097
13	5.3384	-84.3144	-204.0903	-3.3322	-19.5056	-84.0465	-0.1617	0.6856	6.4098
14	5.3383	-84.3144	-204.0902	-3.3322	-19.5057	-84.0465	-0.1616	0.6856	6.4098
15	5.3384	-84.3144	-204.0902	-3.3322	-19.5056	-84.0465	-0.1617	0.6856	6.4098

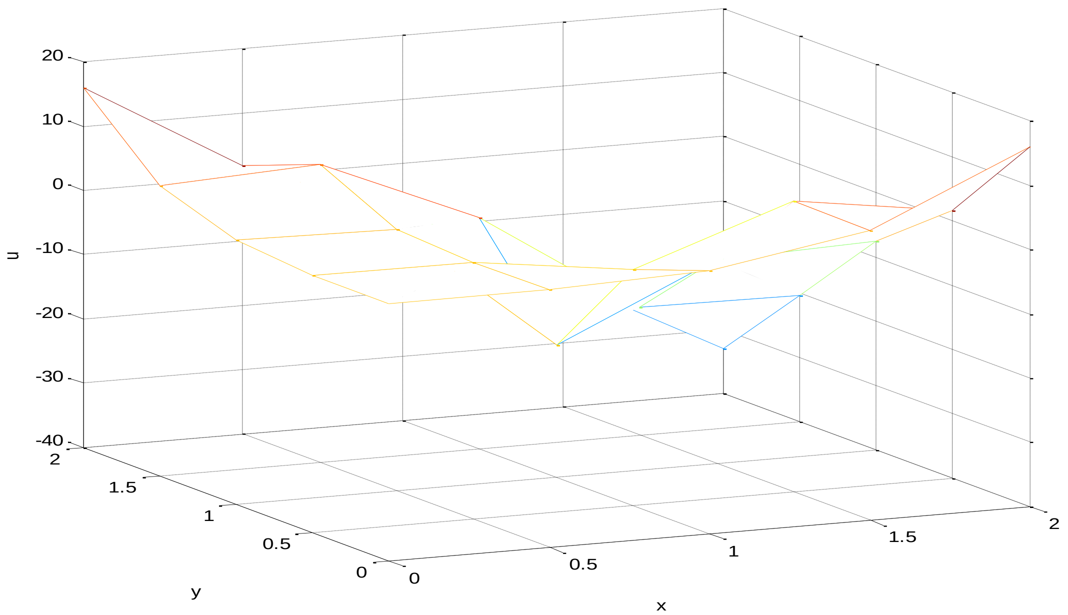

Figure 1 The approximate solution $u = u(x,y)$ of example 1

Example 2

Use GS and SOR iterative methods to compute the approximate solution for Poisson's equation

$$\nabla^2 u \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = (x^2 + y^2)e^{xy} \quad for\ all\ (x,y) \in \Omega$$

$$\Omega = \{(x,y): 0 < x < 2\ , 0 < y < 0.8\}$$

at the boundary values:

$$u(x,0) = 1, \quad u(x,0.8) = e^{0.8x} \quad for\ all \quad 0 \le x \le 2.$$

$$u(0,y) = 1, \quad u(2,y) = e^{2y} \quad for\ all \quad 0 \le y \le 0.8.$$

use $h = 0.4,\ k = 0.2$

Solution

The finite difference equation is

$$10u_{i,j} - u_{i+1,j} - u_{i-1,j} - 4u_{i,j-1} - 4u_{i,j+1} = -0.16f_{i,j}$$

For each $i = 1,2,3,4 \quad j = 1,2,3$

$$p_1 \quad : \quad 10u_1 - u_2 - 4u_5 = u_{0,3} + 4u_{1,4} - 0.16f_{1,3}$$

$$p_2 \quad : \quad 10u_2 - u_1 - u_3 - 4u_6 = 4u_{2,4} \ - 0.16f_{2,3}$$

$$p_3 \quad : \quad 10u_3 - u_2 - u_4 - 4u_7 = 4u_{3,4} - 0.16f_{3,3}$$

$$p_4 \quad : \quad 10u_4 - u_3 - 4u_8 = 4u_{4,4} + u_{5,4} - 0.16f_{4,3}$$

$$p_5 \quad : \quad 10u_5 - 4u_1 - u_6 - 4u_9 = u_{0,2} - 0.16f_{1,2}$$

$$p_6 \quad : \quad 10u_6 - 4u_2 - u_5 - u_7 - 4u_{10} = -0.16f_{2,2}$$

$$p_7 \quad : \quad 10u_7 - 4u_3 - u_6 - u_8 - 4u_{11} = \ -0.16f_{3,2}$$

$$p_8 \quad : \quad 10u_8 - 4u_4 - u_7 - 4u_{12} = u_{5,2} \ - 0.16f_{4,2}$$

$$p_9 \quad : \quad 10u_9 - 4u_5 - u_{10} = 4u_{1,0} + u_{0,1} - 0.16f_{1,1}$$

$$p_{10} \quad : \quad 10u_{10} - 4u_6 - u_9 - u_{11} = 4u_{2,0} - 0.16f_{2,1}$$

$$p_{11} \quad : \quad 10u_{11} - 4u_7 - u_{10} - u_{12} = 4u_{3,0} - 0.16f_{3,1}$$

$$p_{12} \quad : \quad 10u_{12} - 4u_8 - u_{11} = 4u_{4,0} + u_{5,1} - 0.16f_{4,1}$$

The boundary conditions are:

$$u_{0,j} = g(x_0, y_j) = 1 \quad j = 0,1, \dots, 4$$

$$u_{i,0} = g(x_i, 0) = 1 \quad i = 1, \dots, 3$$

$$u_{4,j} = g(x_4, y_j) = e^{2y_j} \quad j = 0,1, \dots, 4$$

$$u_{i,4} = g(x_i, y_4) = e^{0.8x_i} \quad i = 1, \dots, 3$$

The linear system to this example has the form:

$$\begin{bmatrix} 10 & -1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 10 & -1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 10 & -1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 10 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 10 & -1 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & -1 & 10 & -1 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & -1 & 10 & -1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & -1 & 10 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 10 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -1 & 10 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -1 & 10 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -1 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 6.4026 \\ 3.5344 \\ 9.8551 \\ 18.1192 \\ 0.9399 \\ -0.1763 \\ -0.4137 \\ 1.4002 \\ 4.9653 \\ 3.8723 \\ 3.6990 \\ 4.9189 \end{bmatrix}$$

The result for GS iterative method is in the following tables:

Table 3 The approximate solution of Poisson's Equation using GS method

T	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0.6403	0.4175	1.0273	1.9146	0.3501	0.1844	0.3880	0.9447	0.6366	0.5246	0.5776	0.9275
2	0.8220	0.6121	1.3934	2.3291	0.6959	0.5455	0.8960	1.5323	0.8273	0.7459	0.8956	1.1944
3	0.9798	0.8089	1.6577	2.5906	0.8714	0.7810	1.2113	1.7751	0.9197	0.8812	1.0620	1.3081
4	1.0697	0.9386	1.8230	2.7043	0.9679	0.9282	1.3829	1.8833	0.9718	0.9619	1.1501	1.3602
5	1.1213	1.0191	1.9110	2.7563	1.0240	1.0155	1.4729	1.9339	1.0023	1.0087	1.1960	1.3851
6	1.1518	1.0659	1.9569	2.7812	1.0572	1.0652	1.5197	1.9585	1.0203	1.0349	1.2198	1.3973
7	1.1697	1.0922	1.9807	2.7934	1.0765	1.0928	1.5440	1.9707	1.0306	1.0494	1.2322	1.4034
8	1.1801	1.1067	1.9931	2.7995	1.0876	1.1079	1.5566	1.9768	1.0365	1.0573	1.2386	1.4065
9	1.1859	1.1145	1.9995	2.8026	1.0938	1.1161	1.5632	1.9800	1.0398	1.0615	1.2420	1.4081
10	1.1892	1.1188	2.0029	2.8042	1.0972	1.1205	1.5666	1.9816	1.0416	1.0638	1.2437	1.4089
11	1.1910	1.1210	2.0047	2.8050	1.0991	1.1229	1.5684	1.9824	1.0425	1.0650	1.2447	1.4093
12	1.1920	1.1223	2.0056	2.8055	1.1001	1.1241	1.5694	1.9829	1.0431	1.0657	1.2452	1.4096
13	1.1925	1.1229	2.0061	2.8057	1.1006	1.1248	1.5699	1.9831	1.0434	1.0660	1.2454	1.4097
14	1.1928	1.1233	2.0064	2.8058	1.1009	1.1252	1.5702	1.9832	1.0435	1.0662	1.2456	1.4097
15	1.1930	1.1234	2.0065	2.8059	1.1011	1.1253	1.5703	1.9833	1.0436	1.0663	1.2456	1.4098
16	1.1930	1.1235	2.0066	2.8059	1.1012	1.1254	1.5704	1.9833	1.0436	1.0663	1.2457	1.4098
17	1.1931	1.1236	2.0066	2.8059	1.1012	1.1255	1.5704	1.9833	1.0437	1.0664	1.2457	1.4098
18	1.1931	1.1236	2.0066	2.8059	1.1012	1.1255	1.5704	1.9833	1.0437	1.0664	1.2457	1.4098
19	1.1931	1.1236	2.0066	2.8059	1.1013	1.1255	1.5705	1.9834	1.0437	1.0664	1.2457	1.4098

The result for SOR iterative method is in the following tables:

Table 4 The approximate solution of Poisson's Equation using SOR method

T	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0.7750	0.5217	1.2561	2.3454	0.4891	0.2905	0.5933	1.3770	0.8379	0.7108	0.8211	1.3616
2	0.9118	0.7211	1.5870	2.5585	0.8932	0.7908	1.2535	1.9295	0.9432	0.9156	1.1575	1.3832
3	1.1029	0.9846	1.8947	2.8184	1.0122	1.0065	1.5194	1.9817	1.0034	1.0250	1.2313	1.4128
4	1.1522	1.0768	2.0013	2.8018	1.0663	1.0974	1.5680	1.9829	1.0302	1.0581	1.2469	1.4091
5	1.1791	1.1176	2.0054	2.8064	1.0919	1.1231	1.5706	1.9834	1.0410	1.0668	1.2455	1.4099
6	1.1908	1.1233	2.0070	2.8059	1.1005	1.1260	1.5706	1.9834	1.0439	1.0665	1.2458	1.4098
7	1.1932	1.1240	2.0067	2.8060	1.1016	1.1258	1.5705	1.9834	1.0438	1.0665	1.2457	1.4098
8	1.1933	1.1237	2.0067	2.8059	1.1014	1.1256	1.5705	1.9834	1.0437	1.0664	1.2457	1.4098
9	1.1932	1.1237	2.0067	2.8059	1.1013	1.1256	1.5705	1.9834	1.0437	1.0664	1.2457	1.4098
10	1.1931	1.1236	2.0067	2.8059	1.1013	1.1256	1.5705	1.9834	1.0437	1.0664	1.2457	1.4098

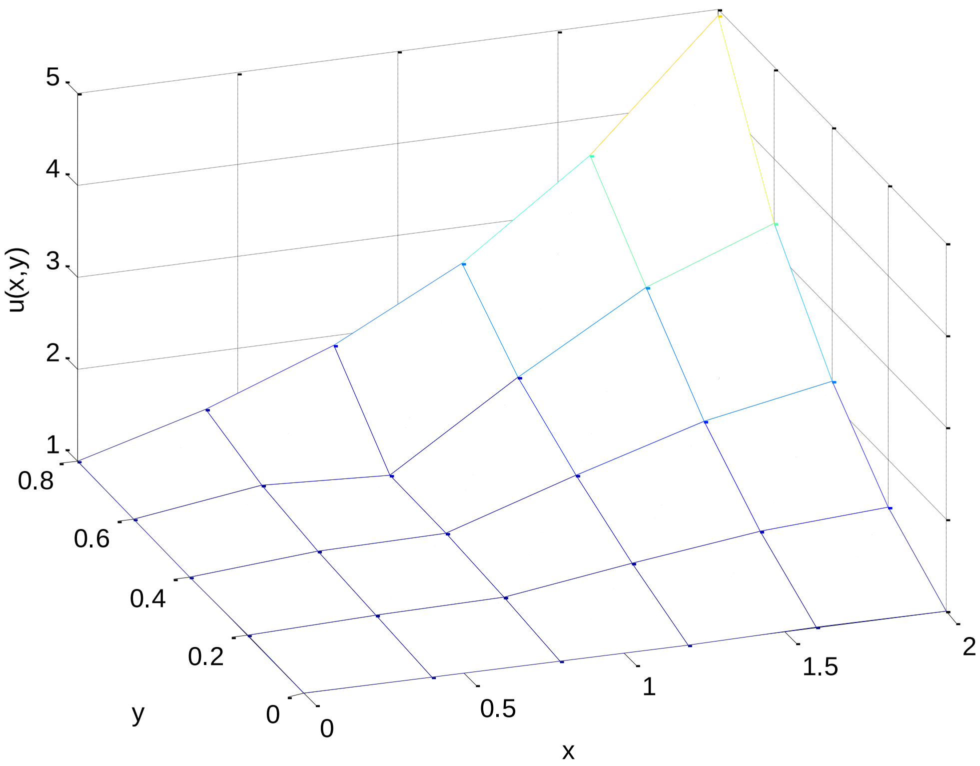

Figure 2 The approximate solution $u = u(x, y)$ of example 2

Conclusion.

A linear system solved in this study by the iterative methods GS and SOR. Such methods were used successfully to obtain the approximate solutions of a linear system resulting from applying the finite difference method of Laplace's and Poisson's Equations. The FDM is an efficient technique for solving partial differential equations

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تطبيق طريقة الفروق المحدودة لحل المعادلات التفاضلية الجزئية الناقصية باستعمال الطرق التكرارية [GS], [SOR]

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الملخص.

في هذه الورقة إيجاد حل تقريبي للنظام الخطي Au = B. بواسطة الطرق التكرارية [[GS و[SOR]، تم اشتقاق هذا النظام من المعادلة التفاضلية الجزئية الناقصية من خلال طريقة الفروق المحدودة. وتطبيق هذه الطرق على مثالين مختلفين. وأظهرت النتائج بأن التقارب إلى الحل التقريبي لهم ومع اختيار خاص لـ ω لطريقة [SOR] التكرارية ومقارنة النتائج لهما.

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